SUMMIT

Problem of the Week
Problem D and Solution
Sum It

Problem

A sequence of non-negative integers is formed in the following way: the first two terms of the sequence are defined, and then each term after the second term is the sum of all previous terms in the sequence. For example, if the first two terms of the sequence were 2 and 8, the next four terms of the sequence would be $10, 20, 40$ and 80.

A sequence is formed as described above such that the first term is 3 and some other term in the sequence is 3072. How many such sequences are there?

Solution

We know how to construct the sequence and we know that it starts with first term 3, but where is the term whose value is 3072?

Can the second term be 3072?

If the first two terms are 3 and 3072, then the third term would be
$3 + 3072 = 3075$.
The fourth term would be $\mathbf{3} + \mathbf{3072} + 3075 = \mathbf{3075} + 3075 = 2(3075) = 6150$.
The fifth term would be
$\mathbf{3} + \mathbf{3072} + \mathbf{3075} + 6150 = \mathbf{6150} + 6150 = 2(6150) = 12\,300$.

We see that we can determine any term beyond the third term by summing all of the previous terms or we can simply double the term immediately before the required term, since that term is the sum of all the preceding terms. (This also means that if any term after the third term is known, then the preceding term is half the value of that term.)

Therefore, there is a sequence with second term 3072. The first 6 terms of this sequence are $3, 3072, 3075, 6150, 12\,300, 24\,600$.

Can the third term be 3072?

Yes, since the third term is the sum of the first two terms and the first term is 3, then the second term is $3072 - 3 = 3069$ and the first 6 terms of the sequence are $3,\ 3069,\ 3072,\ 6144,\ 12\,288,\ 24\,576$.

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Can the fourth term be 3072?

Yes, since the fourth term is even, then we can determine the third term to be half of the fourth term, $3072 \div 2 = 1536$, and the second term is $1536 - 3 = 1533$. This sequence starts $3,\ 1533,\ 1536,\ 3072,\ 6144,\ 12\,288$.

Can the fifth term be 3072?

To get from the fifth term to the third term we would divide by 2 twice, or we could divide by 4. If the resulting third term is a non-negative integer greater than or equal to 3, the sequence exists. The third term is $3072 \div 4 = 768$ and the second term would be $768 - 3 = 765$. Yes the sequence exists and it starts $3,\ 765,\ 768,\ 1536,\ 3072,\ 6144$.

We could continue in this way until we discover all possible sequences that are formed according to the given rules with first term 3 and 3072 somewhere in the sequence. However, if we look at the prime factorization of 3072 we see that the highest power of 2 that divides 3072 is 1024 since $3072 = 2^{10} \times 3$. In fact, dividing 3072 by 1024 would produce a third term that would be 3. The second term would then be 0, a non-negative integer, and the resulting sequence would be $3, 0, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536, 3072, 6144, \cdots$.

If we divide 3072 by any integral power of 2 from $2^0 = 1$ to $2^{10} = 1024$, the resulting third term would be an integer greater than or equal to 3, and 3072 would appear in each of these sequences. There are 11 such sequences. The number 3072 would appear somewhere from term 3 to term 13 in the acceptable sequence. However, 3072 can also appear as the second term so there are a total of 12 possible sequences.

Could 3072 be the fourteenth term? From the fourteenth term to the third term we would need to divide 3072 by 2^{11}. The resulting third term would be $\frac{3}{2}$. This is not a non-negative integer and so the sequence is not possible.

Therefore, there are a total of 12 such sequences.